Θεωρία δυναμό περιστρεφόμενων τυρβωδών ροών

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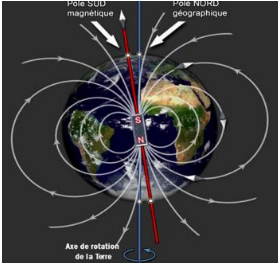
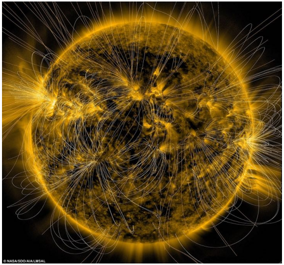
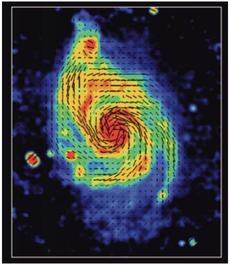
Natural dynamos

The existence of planetary, stellar and galactic magnetic fields is attributed to the dynamo action



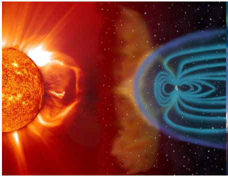


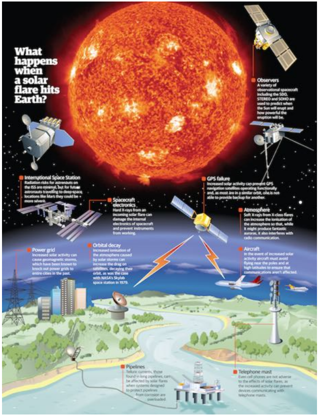


The mechanism by which a background turbulent flow spontaneously generates a magnetic field

Why are magnetic fields important?

- Impact of solar wind on the planet's magnetosphere → magnetic storms
- Geomagnetic storms: observed as aurorae by the naked eye
- Magnetic storms can cause:
  1. interruptions to radio communications and GPS
  2. disruption to power grids
  3. damage to space-crafts
  4. extinction of certain species that use magnetoception





Dynamo problem

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \eta \boldsymbol{\nabla}^2 \mathbf{B}$$

$$\partial_t \mathbf{B} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{B} = \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{u} + \frac{1}{Rm} \boldsymbol{\nabla}^2 \mathbf{B}$$

- $Rm = \frac{|\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})|}{|\eta \boldsymbol{\nabla}^2 \mathbf{B}|} = UL/\eta$
- The induction equation is linear in $\mathbf{B}$
- It admits solutions of the form $\mathbf{B} = \mathbf{b}(\boldsymbol{x}) \exp(\lambda t)$
- The induction equation becomes an eigenvalue problem with $\lambda = \gamma + i\omega$
- For a given $\mathbf{u}$ we have the following solutions
  - $\gamma < 0$: non-dynamo
  - $\gamma > 0$: kinematic dynamo - Rm_c

Anti-dynamo theorems

Cowling's Theorem (1934)

- Axisymmetric magnetic fields cannot be generated via dynamo action
- Field must be inherently 3D for $\gamma > 0$

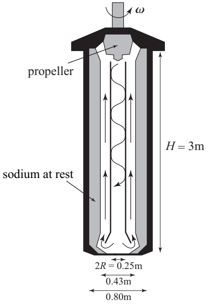
Zel'dovich Theorem (1957)

- Planar velocity fields (2D flow) are not capable of sustaining dynamo action

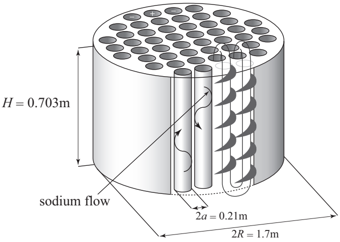
$\therefore$ All analytical and numerical calculations will have to be 3D

Laboratory dynamos

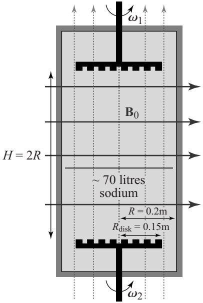
- Since 1960s several experimental groups try to reproduce the dynamo instability in the laboratory using liquid metals
- However, so far, unconstrained dynamos driven just by turbulent flows have not been achieved in the laboratory!!!
- Successful experimental dynamos rely either in constraining the flow (Riga and Karlsruhe) or using ferromagnetic materials (VKS).



Riga



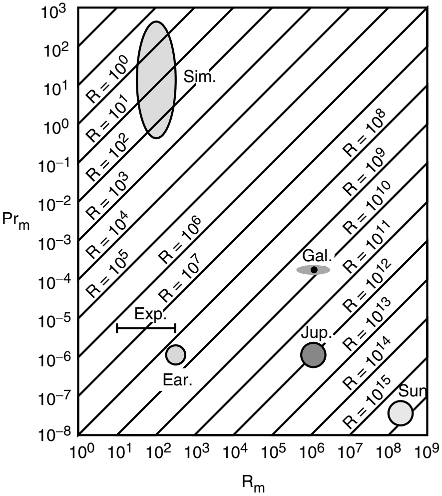
Karlsruhe



VKS - Cadarache

The challenge for liquid-metal dynamos

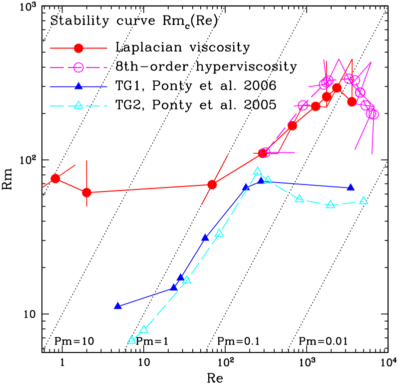
- Liquid Sodium at $T = 393\,K$
  $\nu \sim 10^{-7}\,m^2/s$, $\eta \sim 10^{-1}\,m^2/s$
- Magnetic Prandtl number
  $Pm = \nu/\eta = Rm/Re \ll 1$
- Kinetic Reynolds number
  $Re = UL/\nu \gg 1$
- Energy injection rate
  $\epsilon \propto Re^3$
- The dynamo onset is extremely costly to reach in the laboratory



Fauve & Lathrop, FluidDynAstoGeo (2005)

Numerical dynamos

- Degrees of freedom in DNS $N \equiv (L/\ell_d)^3 \propto Re^{9/4}$ with $\ell_d \propto (\nu^3/\epsilon)^{1/4}$
- The value of Rm_c increases monotonically for values of $Pm \sim 1$
- Turbulent fluctuations prevent the dynamo instability
- For high Re a finite value of Rm_c was reached independent of Re, i.e. $Rm_c^{turb} \equiv \lim_{Re\to\infty} Rm_c$



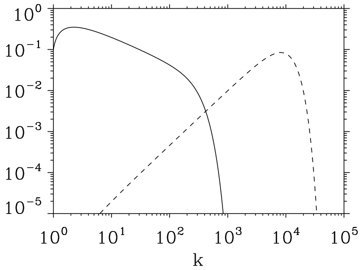
Iskakov et al., PRL (2007)

Turbulent dynamos

- Natural dynamos are highly turbulent and so inherently multi-scale

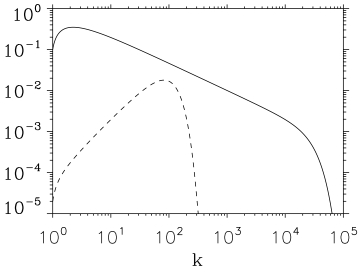
$Pm \gg 1$

Galaxies



$Pm \ll 1$

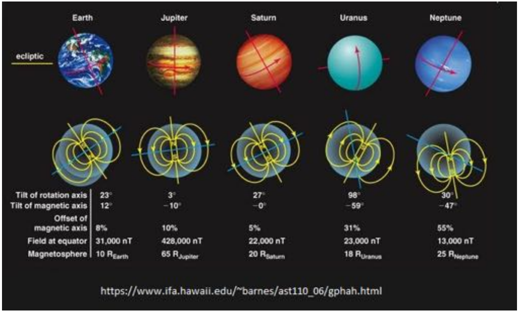
Planets, stars, liquid-metal experiments



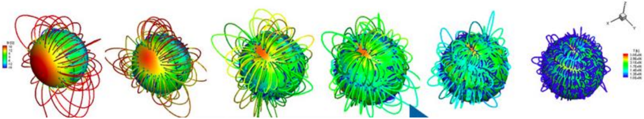
- $Pm \gg 1$: no problem except that the field is generated on very small scales
- $Pm \ll 1$: the magnetic field dissipates in the inertial range of the turbulence
- Note that it is harder to drive a dynamo with a rough velocity than with a smooth velocity

Rotation of planets & stars

Rotation determines the main characteristics of the resulting flows and magnetic fields of planets and stars



Courtesy by NASA



Courtesy by C. Garraffo (Harvard)

Rotating MHD equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P - 2\boldsymbol{\Omega}\times\mathbf{u} + \nu\boldsymbol{\nabla}^2\mathbf{u} + \mathbf{f}$$
$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \eta\boldsymbol{\nabla}^2\mathbf{B}$$

- $\boldsymbol{\Omega} = \Omega\hat{\mathbf{e}}_z$
- Non-helical cellular flow at $k_f L_{box} = 4$
  $\mathbf{f} = f_0(\cos(k_f y), \sin(k_f x), \cos(k_f y) + \sin(k_f x))$
- The non-dimensional parameters $Re = \frac{u}{k_f \nu}$, $Rm = \frac{u}{k_f \eta}$, $Ro = \frac{u k_f}{2\Omega}$
  with $u = (\epsilon/k_f)^{1/3}$ and $\epsilon \equiv \langle \mathbf{u}\cdot\mathbf{f}\rangle$
- Power required for the dynamo onset $\epsilon_c \propto (Rm_c^{turb})^3$
  with $Rm_c^{turb} \equiv \lim_{Re\to\infty} Rm_c$

We are interested in the following limits:

- $Re \gg 1$ limit (or $Pm \ll 1$): we use hyperviscosity $\boldsymbol{\nabla}^8$ for $\mathbf{u}$ only
- $Ro \ll 1$ limit: we use an asymptotic quasi-2D model

Fast rotating limit $Ro \ll 1$

In this limit, $\mathbf{u}$ becomes invariant along the axis of rotation

$$\begin{aligned} \partial_t \mathbf{u}_{2D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{u}_{2D} &= -\nabla p + \nu \nabla^2 \mathbf{u}_{2D} + \mathbf{f}_{2D} \\ \partial_t u_z + \mathbf{u}_{2D} \cdot \nabla u_z &= \qquad\qquad \nu \nabla^2 u_z + f_z \end{aligned}$$

with $\mathbf{u}_{2D} = \boldsymbol{\nabla} \times \psi \hat{\mathbf{e}}_z$.

Due to the invariance of the flow along the z-direction $\mathbf{B} = \mathbf{b}(x, y, t) e^{i k_z z}$

Each k_z-mode evolves independently and the induction equation reads

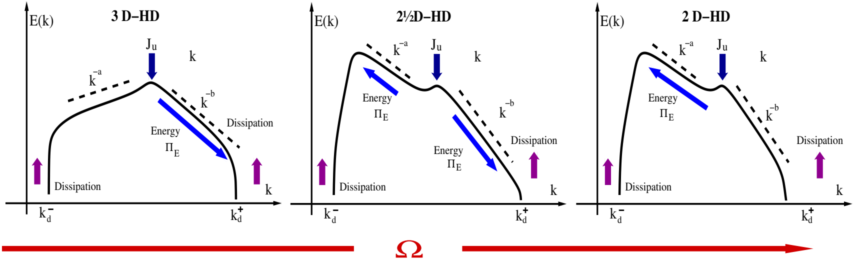
$$\partial_t \mathbf{b} + \mathbf{u}_{2D} \cdot \nabla \mathbf{b} + u_z i k_z \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u}_{2D} + \eta \left(\Delta - k_z^2 \right) \mathbf{b}$$

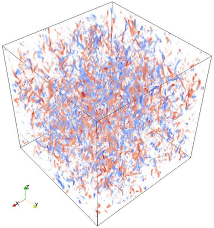
The $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ for each magnetic mode gives

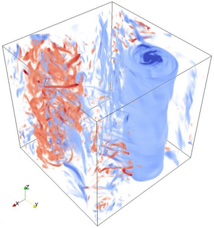
$$\partial_x b_x(x, y, t) + \partial_x b_y(x, y, t) = -i k_z b_z(x, y, t)$$

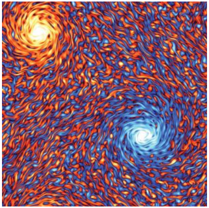
In this limit we follow only the $k_z = 1$ mode that was found to be the most unstable mode (see Seshasayanan & Alexakis, JFM 2016)

Rotating turbulent flows



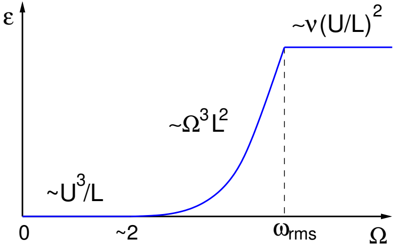


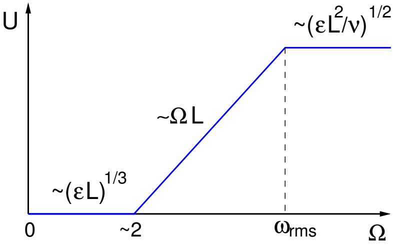


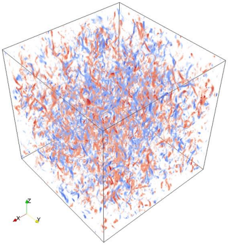


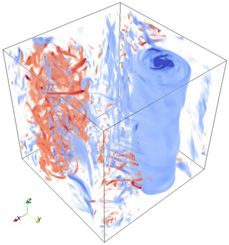
Chan et al., PRE (2012)

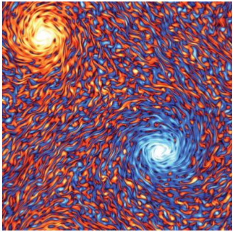
Three regimes









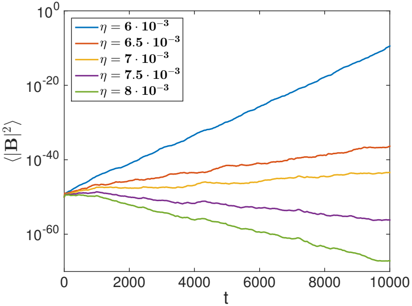


Chan et al., PRE (2012)

Ω

Critical magnetic Reynolds number Rm_c

- To calculate Rm_c we run simulations of the same flow (same Re and Ro) for different values of Rm

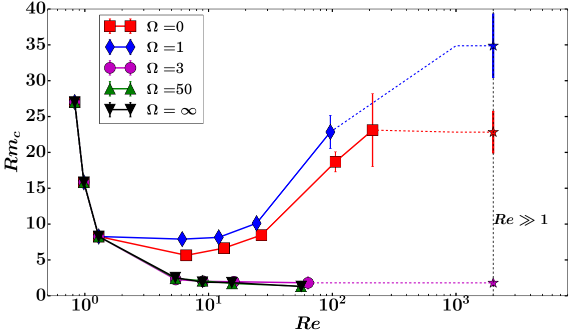


- The growth rate of the magnetic field is computed as

$$\gamma \equiv \lim_{t\to\infty} \frac{1}{2t} \log \frac{\langle|\mathbf{B}|^2(t)\rangle}{\langle|\mathbf{B}|^2(0)\rangle}$$

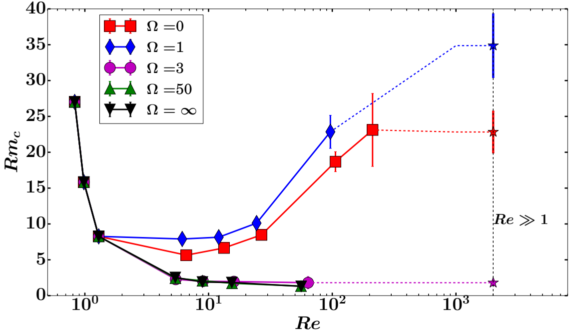
- Rm_c is determined by linear interpolation of the growth-rates between dynamo ($\gamma > 0$) and non-dynamo ($\gamma < 0$) runs

Rm_c as a function of Re for different Ω



- $\Omega = 0$: similar behaviour to other studies of non-rotating dynamos
- $\Omega = 1$: initial hindering effect for the dynamo by rotation
- $\Omega \geq 3$: much lower threshold for the dynamo instability
- $\Omega = 3$: same threshold, implying that the destructive effect of the 3D turbulent fluctuations on dynamo has already disappeared

Power requirements



- The ratio

$$\frac{Rm_c^{turb}|_{\Omega=0}}{Rm_c^{turb}|_{\Omega=3}} \sim 13$$

- So, power consumption reduces by

$$\frac{\epsilon_c|_{\Omega=0}}{\epsilon_c|_{\Omega=3}} \sim 2\cdot 10^3 \quad !!!$$

since $\epsilon_c \propto (Rm_c^{turb})^3$, with $Rm_c^{turb} \equiv \lim_{Re\to\infty} Rm_c$

Practical considerations

Technical constrains limit:

- the size of liquid metal laboratory experiments $L \sim 2m$
- the magnetic diffusivity of liquid sodium $\eta \simeq 10^{-1}$ m^2/s
- the density of liquid sodium $\rho \simeq 10^3$ kg/m^3

Assuming $Rm_c \simeq 50$, energy consumption $\epsilon > 100$ kW

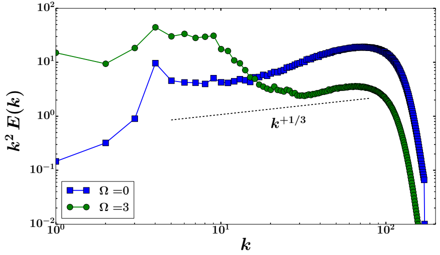
The VKS experiment consumed 300 kW at its peak.

This large ϵ limits dynamo experiments to large industrial size laboratories

A reduction of Rm_c even by a factor of 2, reduces this consumption rate to ~ 10 kW

Such a reduction can make dynamos attainable in small scale laboratories!

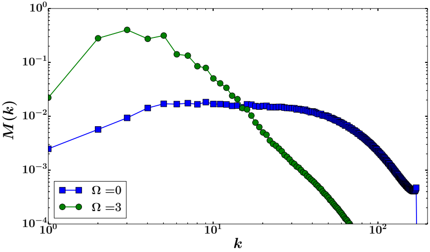
Kinetic energy spectra



- Large enstrophy implies a large stretching rate u_ℓ/ℓ of the magnetic field lines
- $\Omega = 0$: close to Kolmogorov behaviour with $E(k) \propto k^{-5/3}$ with the strongest stretching rate at the small incoherent scales
- $\Omega = 3$: $k^2 E(k)$ decreases with k. At the smallest scales the $k^{1/3}$ starts to form again
    - small scale fluctuations are suppressed
    - the dominant stretching rate is restricted to the large scales

Magnetic energy spectra

Magnetic energy spectra for Rm close to the onset

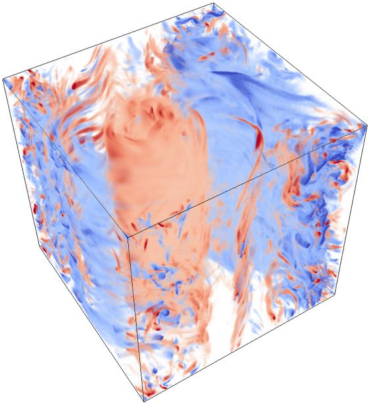


- $\Omega = 0$: spectrum is almost flat with an exponential cut-off
- $\Omega = 3$: spectrum decreases fast with k, and peaks at $(k_f = 3)$, while magnetic energy at $k = 1$ is an order of magnitude smaller

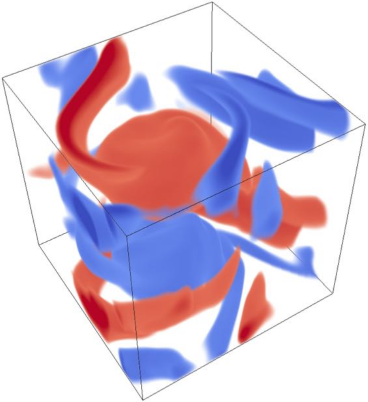
Visualisations

Structures from an unstable eigenmode of the dynamo at $\Omega = 3$

vertical vorticity field $\boldsymbol{\omega}_z$



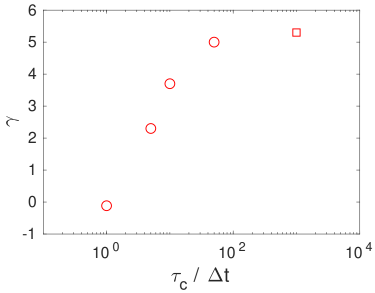
vertical currect field $\boldsymbol{j}_z$



- The dynamo behaves as if it is driven by an organised laminar flow (i.e. high Pm behaviour) even at very large Re (i.e. at low values of Pm).

Dynamo growth rate dependence on coherence time

- We compare dynamos with the same spectra but different coherence time
- We consider the flow with $\Omega = 3$
- We randomise the phases of each Fourier coefficient at different coherence times τ_c
- $\hat{\mathbf{u}}_{new}(\mathbf{k}_\perp) = \hat{\mathbf{u}}(\mathbf{k}_\perp)\exp(i\phi_{k_\perp})$,
  $\phi_{k_\perp}$: random numbers
  $k_\perp = \sqrt{k_x^2 + k_y^2}$



- $\tau_c/\Delta t = \infty$: flow without randomised phases ($\square$)
- $\tau_c/\Delta t = 1$: flow with delta-correlation in time

Conclusions

- Rm_c for a turbulent non-helical dynamo in the $Pm \ll 1$ limit can be significantly reduced if the flow is submitted to global rotation
- Even for moderate rotation rates (i.e. $Ro = 0.2$) the required energy injection rate can be reduced by a factor of more than 10^3
- This suggests a new paradigm to realise liquid metal dynamo experiments in small-scale laboratories
- This strong decrease of Rm_c is due to
  1. the suppression of turbulent fluctuations and
  2. the spatio-temporal organisation of the large scales
- The dynamo growth rate is determined by the long-lived large scale coherent eddies